JOURNAL OF COMBINATORIAL THEORY 38, 93–94 (1985)

Note

Even Edge Colorings of a Graph

NOGA ALON

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

AND

YOSHIMI EGAWA

Department of Applied Mathematics, Science University of Tokyo, Shinjuku-ku, Tokyo, 162 Japan

Communicated by the Managing Editors

Received April 30, 1984

It is shown that the minimum number of colors needed to paint the edges of a graph G so that in every cycle of G there is a nonzero even number of edges of at least one color is $\lceil \log_2 \chi(G) \rceil$.

For a simple graph G, let $\varepsilon(G)$ denote the minimum number l for which there exists a partition of $E(G)$ into l subsets E_i, $1 \leqslant i \leqslant l$, satisfying

$$\begin{aligned}&\text{for any cycle } Z \text{ of } G,\ 0 < |E(Z) \cap E_i| \equiv 0 \pmod 2\\ &\text{for at least one } E_i. \qquad (1)\end{aligned}$$

The number obtained by replacing the condition (1) by

$$\text{each } \langle E_i \rangle \text{ is bipartite} \qquad (2)$$

is denoted by $\varepsilon'(G)$. Acharya [1] conjectured that for any integer $n \geqslant 0$, $\varepsilon(K_m) = n$ for all values of m with $2^{n-1} + 1 \leqslant m \leqslant 2^n$. In this note, we show

PROPOSITION. *If G is a graph with $2^{n-1} + 1 \leqslant \chi(G) \leqslant 2^n$, then $\varepsilon(G) = \varepsilon'(G) = n$, where $\chi(G)$ denotes the chromatic number of G.*

0095-8956/85 $3.00

Proof. Let $E(G)=\bigcup_{i=1}^{l} E_i$ be a partition satisfying (2), and let $V(G)=V_{i1}\cup V_{i2}$ be a bipartition of $\langle E_i\rangle$. Then the partition of $V(G)$ into the $V_{1\lambda_1}\cap\cdots\cap V_{l\lambda_l}$, where $(\lambda_1,\ldots,\lambda_l)$ ranges over $\{1,2\}^l$ yields a proper coloring. Thus $\varepsilon'(G)\geqslant n$.

Now let $V(G)=\bigcup_{\lambda\in\{1,2\}^n} V_\lambda$ be a partition of $V(G)$ into independent subsets. Set

$$E_i=\{e\in E(G)\mid e \text{ joins vertices of } V_\lambda \text{ and } V_\mu$$
$$\text{with } i=\min\{k\mid pr_k(\lambda)\neq pr_k(\mu)\}\},\qquad 1\leqslant i\leqslant n.$$

Then for each cycle Z, $0<|E(Z)\cap E_{j(Z)}|\equiv 0 \pmod 2$, where $j(Z)=\min\{i\mid E(Z)\cap E_i\neq\phi\}$. Thus $\varepsilon(G)\leqslant n$. Since it is clear that $\varepsilon(G)\geqslant\varepsilon'(G)$, this proves the proposition.

Reference

1. B. D. Acharya, Even edge coloring of a graph, *J. Combin. Theory Ser. B* **35** (1983), 78–79.